①

theme

Combinatorial (discrete) structure

ex graph  $\longleftrightarrow$ $\begin{array}{c|cc} & u & v \\ \hline u & 0 & 1 \\ v & 1 & 1 \end{array}$

associate an algebra

* Certain graphs $\xrightarrow[\text{produce}]{}$ algebras already studied in litreature

 $\longrightarrow$ Leavitt algebra

 $\longrightarrow$ toplitz algebra

* study these algebras

theme ①

E $\longleftrightarrow$ $A(E)$
graph — algebra

properties of graph $\longleftrightarrow$ properties of algebra

theme ②

classification

E 

F 

$A(E) \cong A(F)$?

Is there easy to compute invariant

$\vartheta(A(E)) \cong \vartheta(A(F)) \xleftrightarrow{\text{iff}} A(E) \cong A(F)$

E graph $\longrightarrow$ $A(E)$ algebra

various kind of graph $\xrightarrow{\text{create}}$ Similar kind of algebras

Invariant $\longrightarrow$ Commutative monoid $\longleftarrow$ simplied algebraic structure

ex $\mathbb{N} = \{0, 1, 2, \dots\}$

Combinatorial algebras — Chip firing (Sand pile model)

monoid

Symbolic dynamics

We start with Sandpile $\longrightarrow$ monoid $\longrightarrow$ Combinatorial algebra

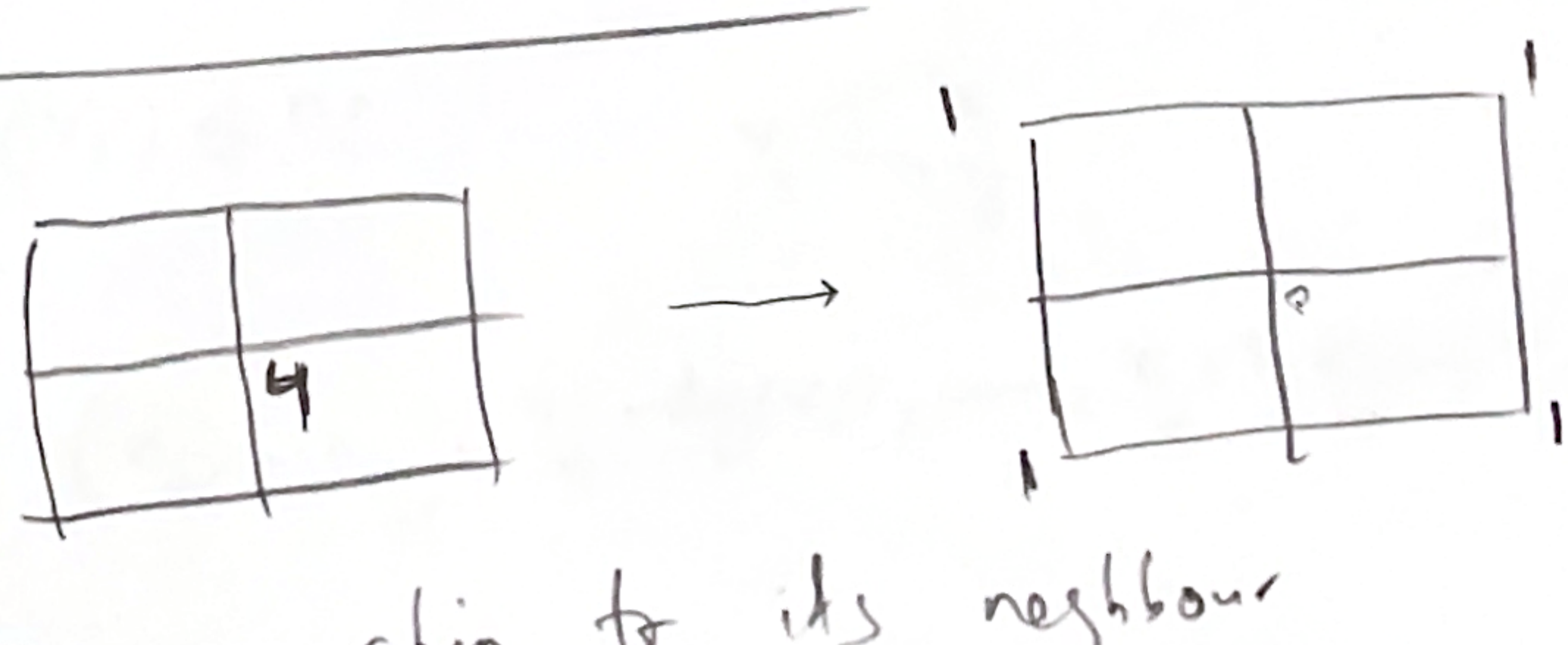
Chip firing (Abelian Sandpile model)

- A system which organized itself with no 'input' from outside
- [either stabilizes or becomes infinite process
- The model uses a "simple local rule" which produces "self-similar" $\boxed{\text{very complex}}$ global patterns.

Sandpile $\longrightarrow$ associate it a mathematical objects

simple local rule $\longrightarrow$ math obj monoid

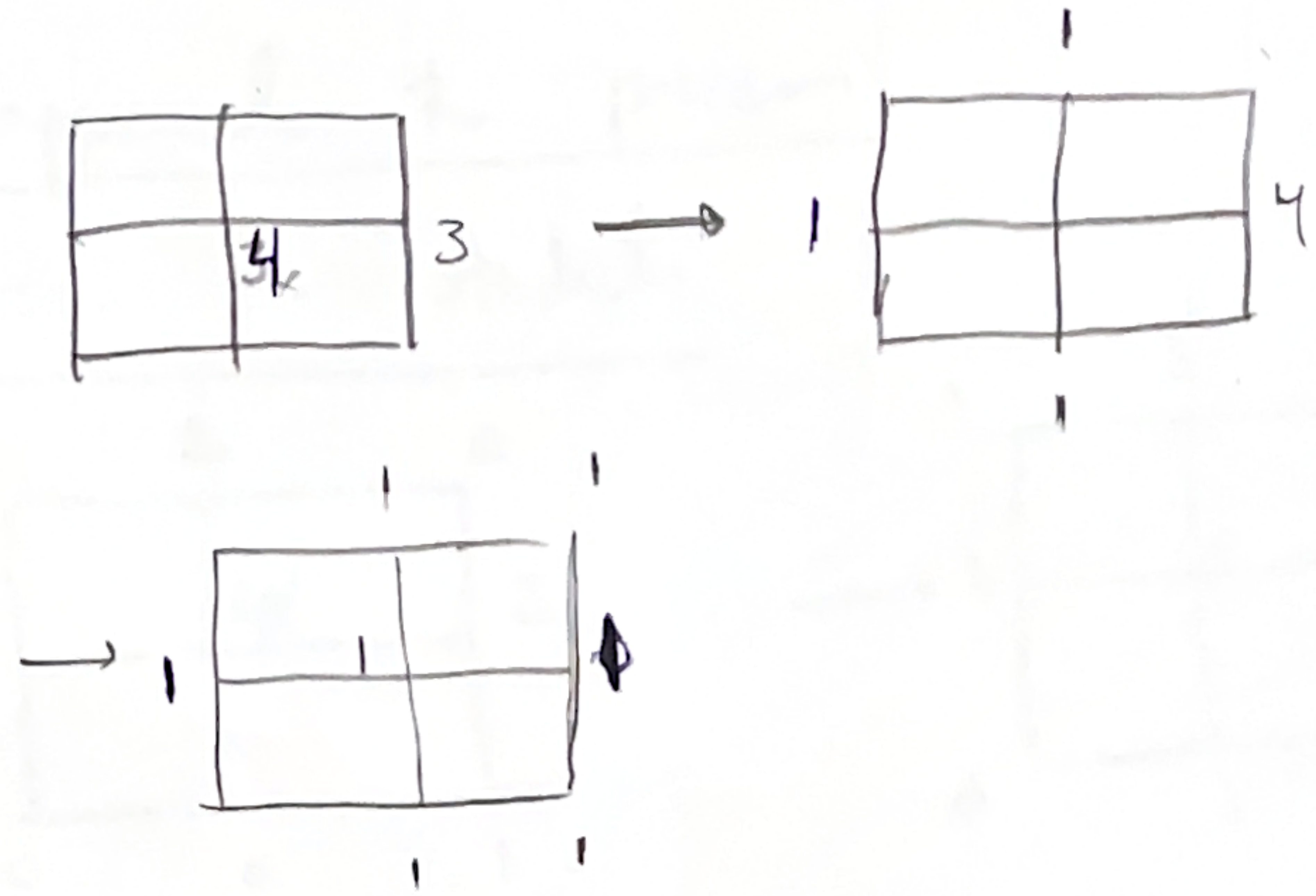
we start with an example



sending a chip to its neighbour

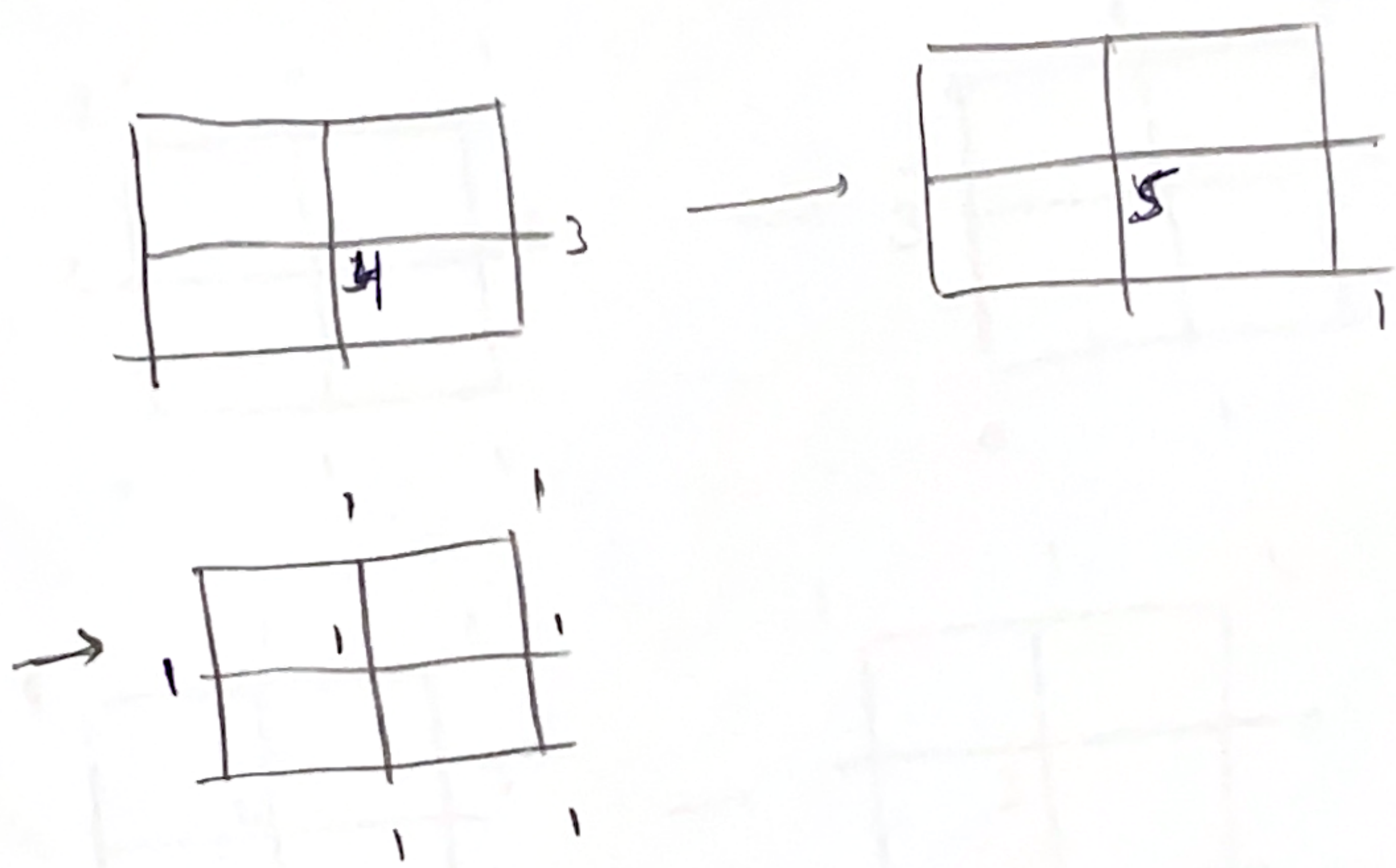
Ex

1



other firing

2



mathematical thinking

$C = (n_{v_1}, n_{v_2}, \ldots, n_{v_n})$

v_i are vertices or nodes

n_{v_i} number of chips on v_i

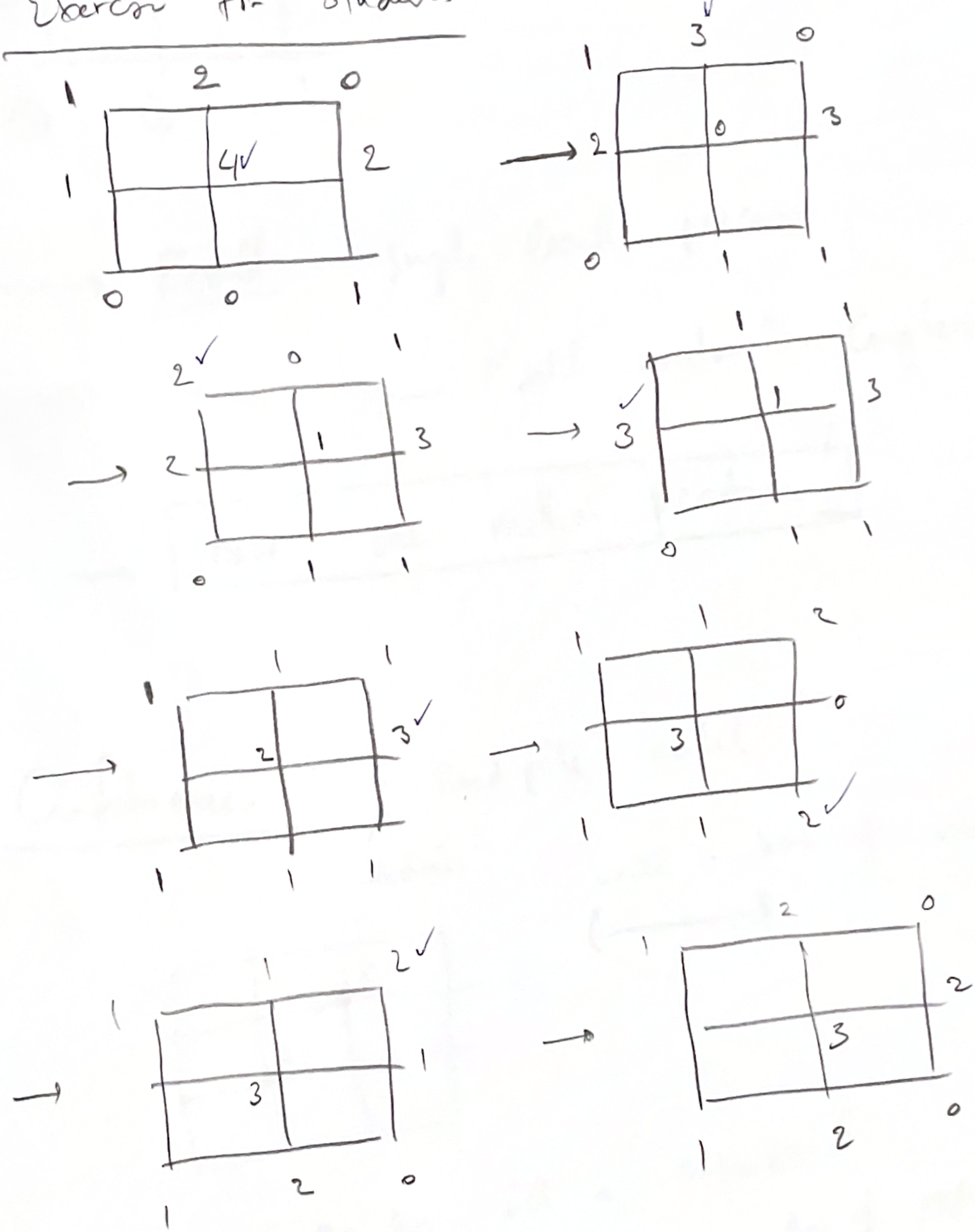
if $\deg(v_i) \leq n_i$

$v_i \to v_k$, $v_i \to v_j$

$C \Longrightarrow (n_{v_1}, \ldots n_{v_{i-1}}, n_{v_i} - \deg(v_i), \ldots, n_k + 1, n_{i+1}, \ldots, n_j + 1, \ldots)$

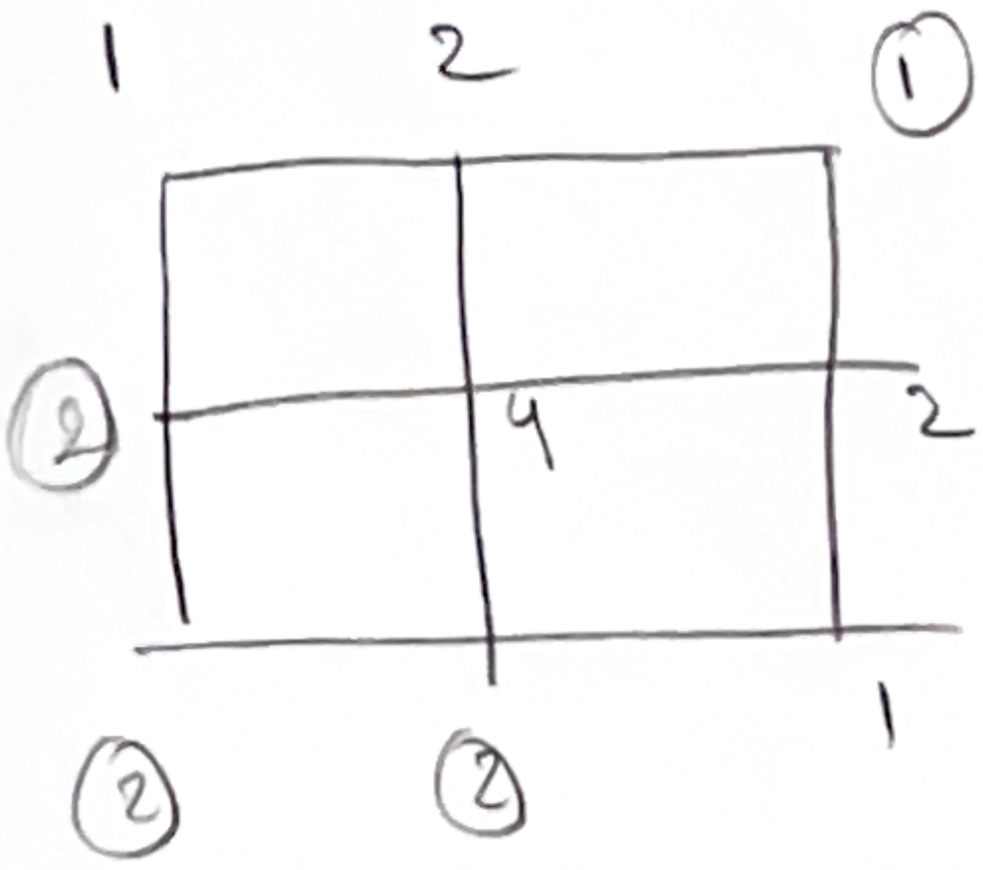
to appreciate the process

Exercise for students



✓ stabilized ✓ unique configuration

✓ 7 steps



would not stabilize -

→ Recall single local process

→ "self similar" complex pattern

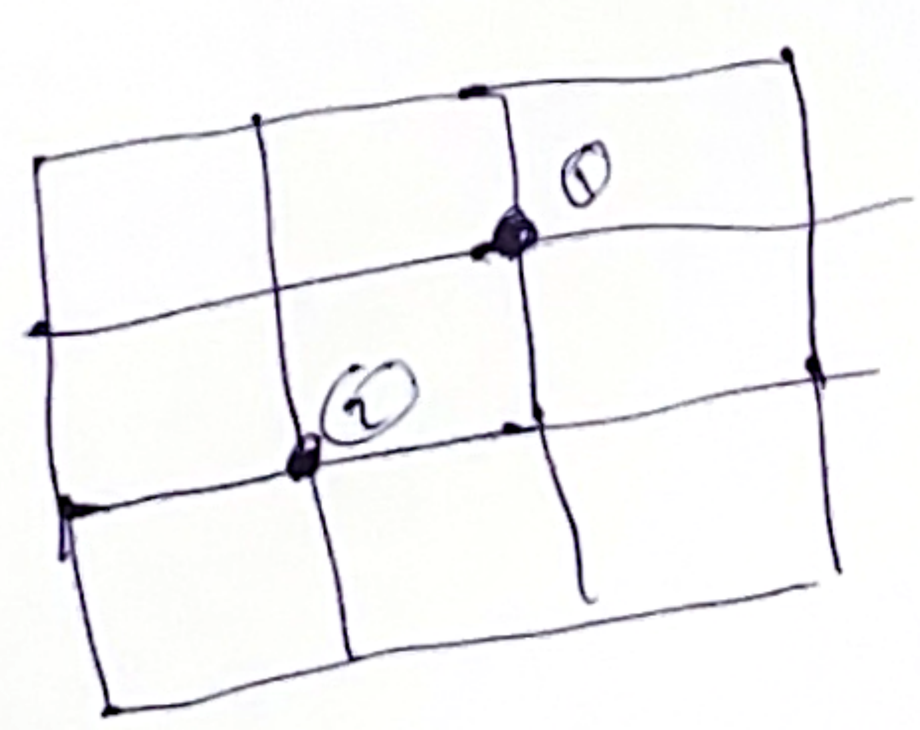
→ show one million picture

Confluence

abelian Sand pile model

write in terms of vector

(———)



M. Newman "On theories with a combinatorial definition of equivalence" Ann of Math 1941

G. Bergman "The diamond lemma for ring theory" Adv. Math 1978

"The main results in this paper are trivial"

$\boxed{\text{Early results}}$

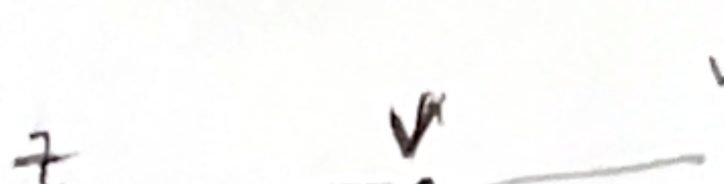
Bjorner-Lovasz, Shor "Chip-firing Games on Graphs"

European J. Combinatorics 1991

Graphs ~~non~~ un directed - simple - connected

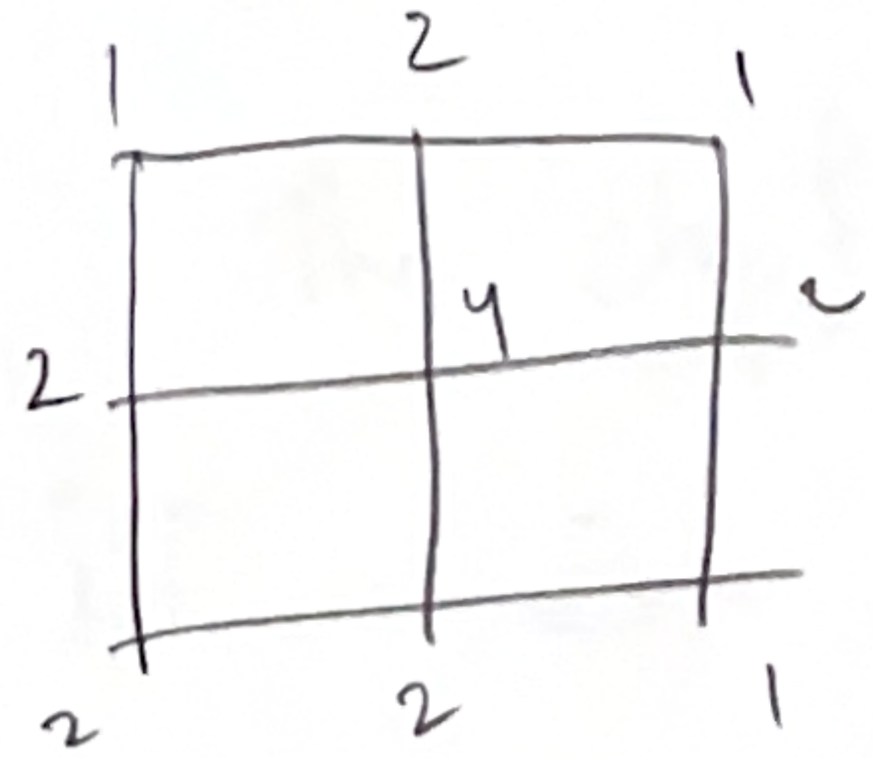
Lemma If a configuration c on a graph does not stabilises then every vertex of the graph fires infinitely many times starting from c.

proof If the procces does not stabises, then $\exists$ vertex v which fires infinite time, So v feed the adjacent vertices



infinite chips so w has to fire infinite times.

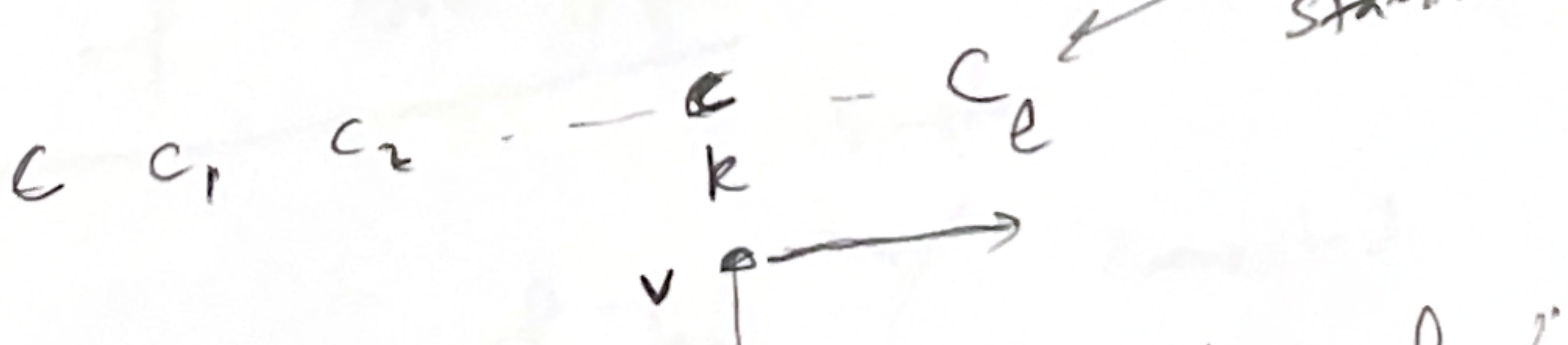
Since the graph is connected, repeating this process all vertices fire infinite times.

So 

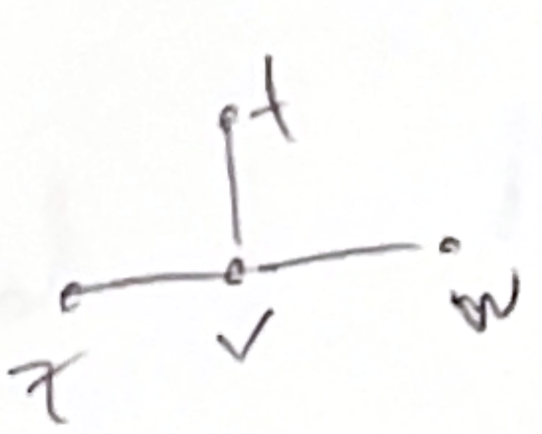
all vertices fires infinite times.

Lemma 2 If from configuration c every vertex can fire at least once, then the chip-firing from c will never stablises.

proof Suppose the configuration c stablises and all vertices fired at least once.



suppose v is the first vertex that stop firing. That means all other vertices firing after configiration C_k.



Then v receives at least deg(v) chips and becomes unstable. Contradiction.

<u>Lemma 3</u> The chipfiring process stablises iff $\exists$ vertices which never fires during the process

<u>proof</u> $\rightarrow$ if all vertices fire $\xrightarrow{\text{Lemma}}$ configuration does not stablises

so the configuration stablises if $\exists$ vertex which does not fire

$\leftarrow$ if a configuration does not ~~st~~stabilise $\rightarrow$ all vertices fire infinite times

---

<u>Main theorem of the paper</u>

<u>Theorem</u> Let E be a finite simple connected graph

$|E^0| = n$, $|E^1| = m$ and c a configuration

such that $N = \sum c_i$ (the number of chips).

① if $N > 2m-n$ then c is nonstable

(chip firing process is infinite)

② if $m \leq N \leq 2m-n$, then

$\exists$ configuration which stabilises and

$\exists$ configuration which does not!

③ $N < m$ then configuration stabilises.

~~Proof~~

Examples

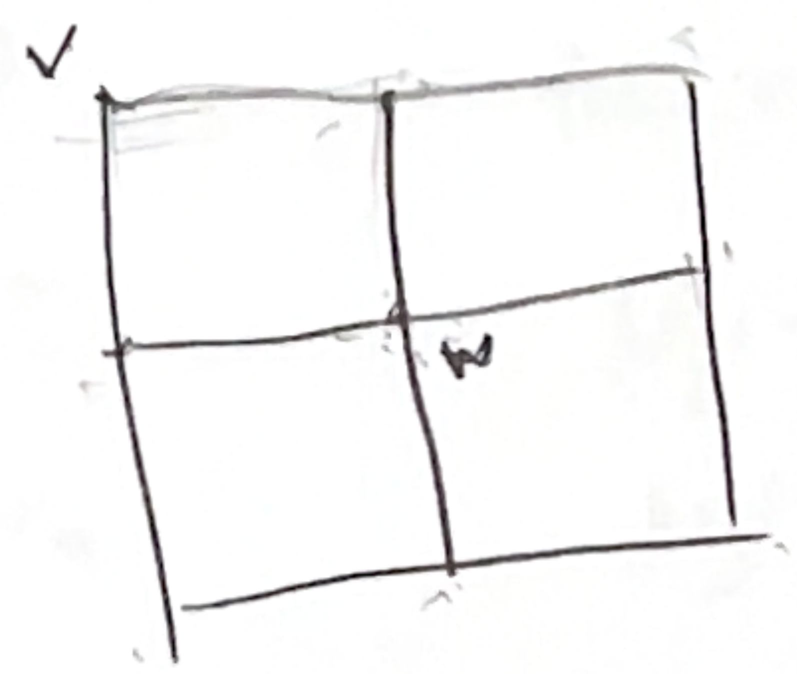
Def E connected graph.

$d(v, w)$ = shortest path between v and w

$\mathrm{diam}(E)$ = largest distance between two vertex

$$\mathrm{diam}(E) = \max \{ d(v,w) \mid v, w \in E \}$$

Ex E



$d(v, w) = 2$
$\mathrm{diam}(E) = 4$

Theorem (Tardos) Let E be a finite $\boxed{\text{simple}}$ undirected connected graph with $|E^0| = n$, $|E^1| = m$, $d = \mathrm{diam}(E)$. Then any configuration which stabilise, will stabilises within $2mnd$ fires.

proof of theorem

1 Fact connected simple graph

$$\sum_{v \in E} \deg(v) = 2m$$

(hand shake Lemma)

the vertices do not fire if

$\deg(v_1)-1$
$\deg(v_2)-1$ chips on them
$\vdots$
$\deg(v_n)-1$

so $$\sum_{v \in E} \deg(v) - n = 2m-n$$

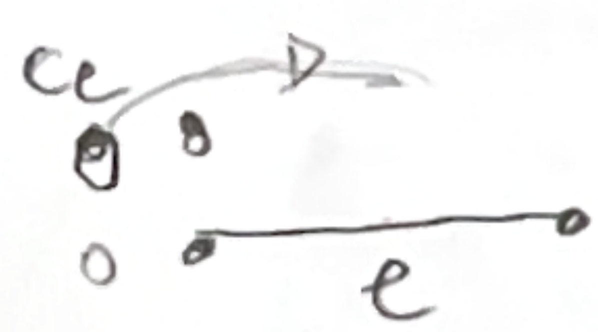
so if $N > 2m-n$ by pigeon hole principle

one vertex has to fire.

2 Exercise session

(3) Suppose $N < m$.

For each edge e mark a chip that fires across e. So in all subsequent firing the chip either does not move or moves along e.



Since $N < m$, there is an edge with no associated chip. So there are two vertices that never fires.

By previous lemma the configuration will stabise

proof of theorem 2   2mnd stablisation

Lemma Configuration with N chips.

for neighbour u and v, the number of times u has fired cannot differ from the number of times that v has fired by more than N. times proof

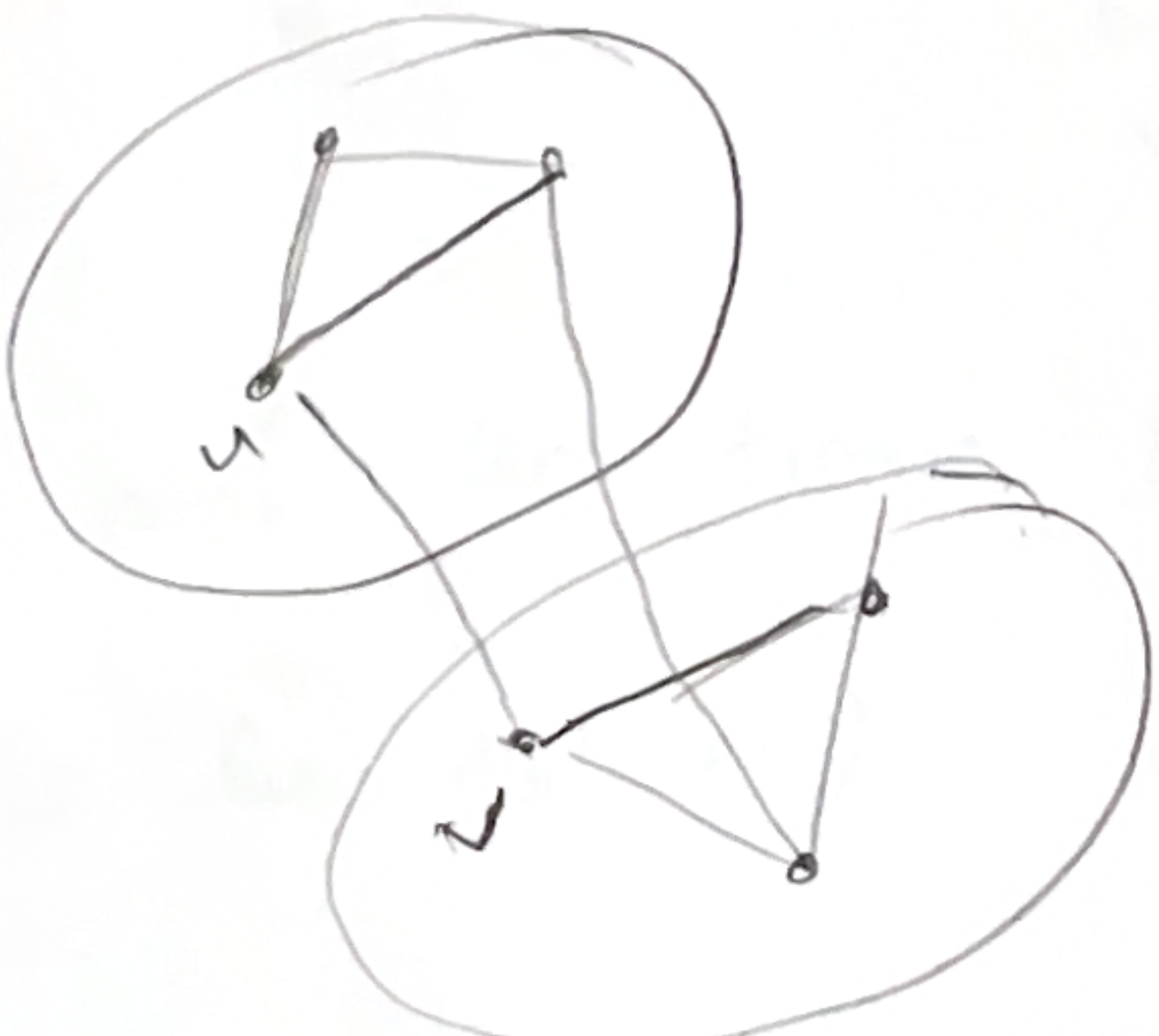
Suppose u has fired a times

v has fired b times

H_a subgraph all vertices fired at most a times

$V \setminus H_a$ the rest of the graph.

H_a



vertices have fire more

Than a times

not empty

Proof of theorem

Suppose the configuration with N chips stabilize.

By previous lemma $\exists v$ which does not fire.

By lemma w can fire at most N fires.

So z can fire at most $2N$ times (The difference between w and z should be less than N). So each vertex can max fire $N \cdot d$. By previous Thm $\underline{N \leq 2m-n} \leq 2m$

so $2md$ and there are n vertice so in total $2mnd$ max firing.

Model with directed graph and sink

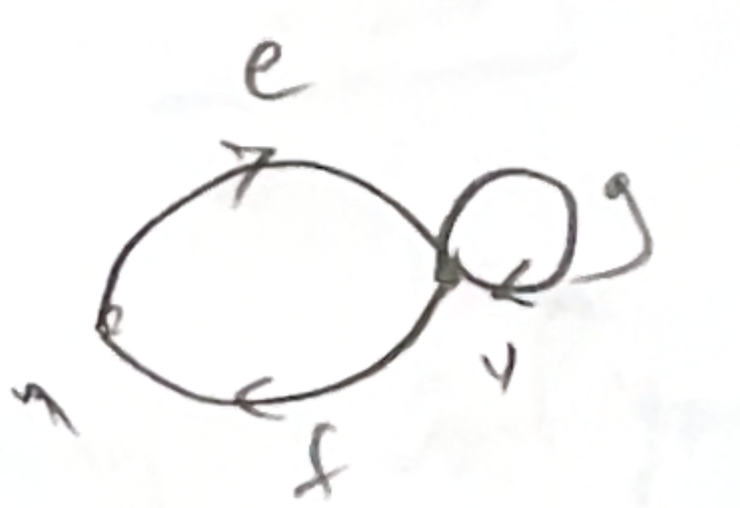
E a directed graph

$E = (E^0, E^1, r, s)$ $|E^0|, |E^1|$ finite

$r: E^1 \to E^0$ (range)

$s: E^1 \to E^0$ (source)

Ex



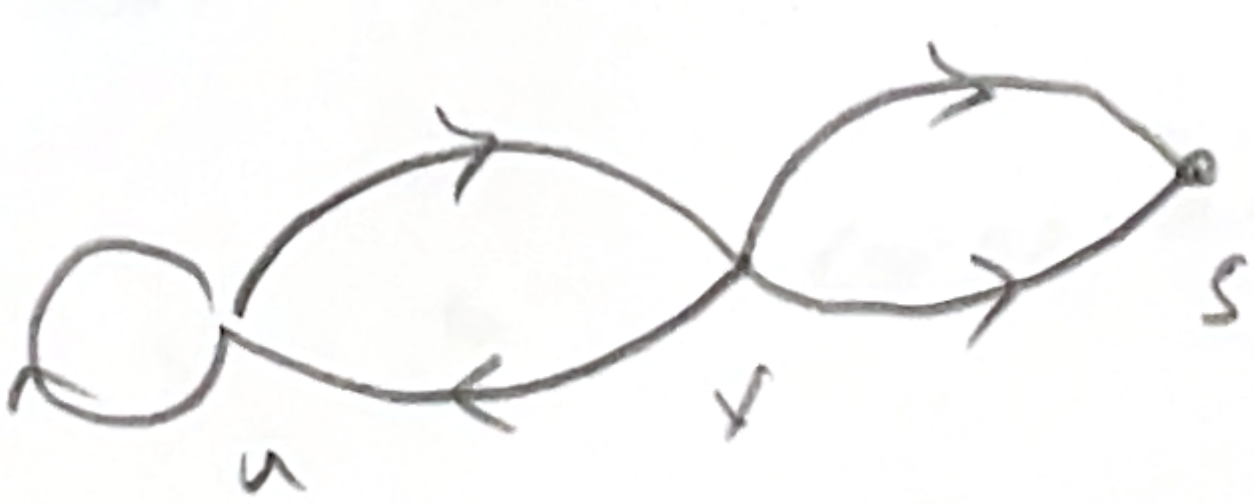
$E^0 = \{u, v\}$

$E^1 = \{e, f, g\}$

$s(e) = u$ $s(g) = r(g) = v$

$r(e) = v$

Def A sandpile graph is a directed graph with a unique sink and all vertices connected to the sink.

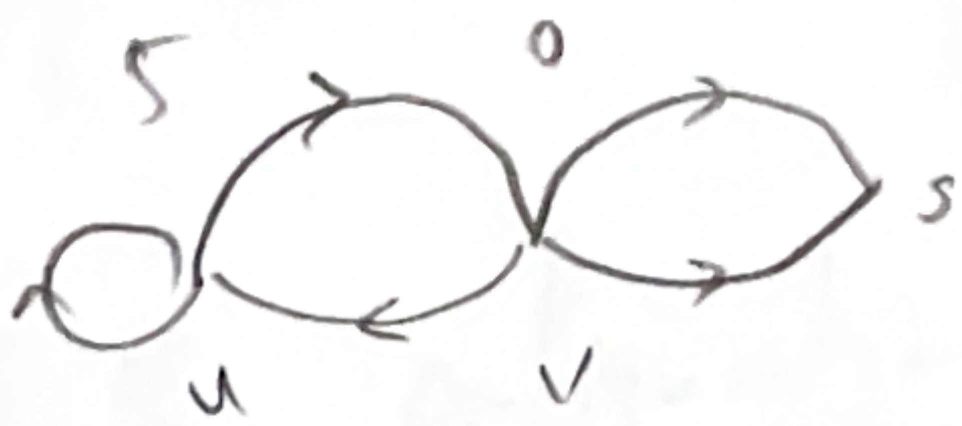


$\deg(v) = |s^{-1}(v)|$ number of edges emit from v (out-degree)

$|r^{-1}(v)|$ number of edges enter to v (in-degree)

<u>model</u> <u>assumption</u> any grain reaches sink disappear.

<u>Ex</u>



$5u = 4u + v = 3u + 2x = \underline{2u} + \underline{3v} = 3u = 2u + v$

$\boxed{= u + 2x}$ stable unique configuration

<u>Lemma</u> Any configuration stablises.

<u>proof</u> Suppose a configuration does not stabilise.

then $\exists$ vertex v which fires infinite times.

$v \to w$ then all $r(s^{-1}(v))$ also fire infinite

times. since v is connected to sink s,

infinite chips must disappear. ✗.

we don't use these terminology

A sequence of topplings (firing) that terminates (give stable configuration) is called an avalanche.

towards assigning a mathematical object

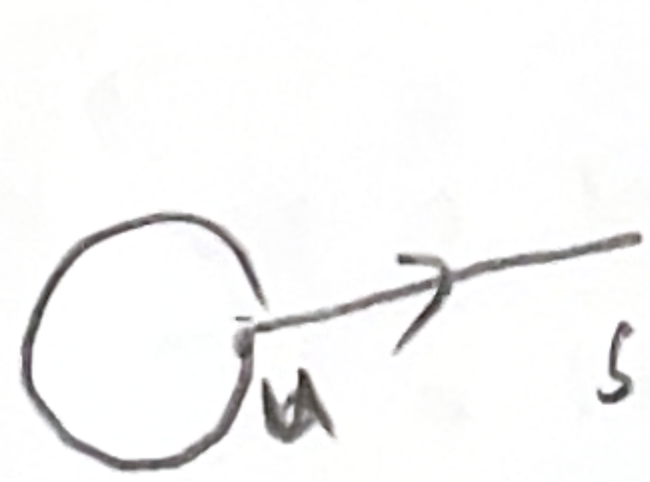
stable configurations

$0u, 1u, 2u, 3u = 0$

	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\leftarrow \{0,1,2\}$

monoid



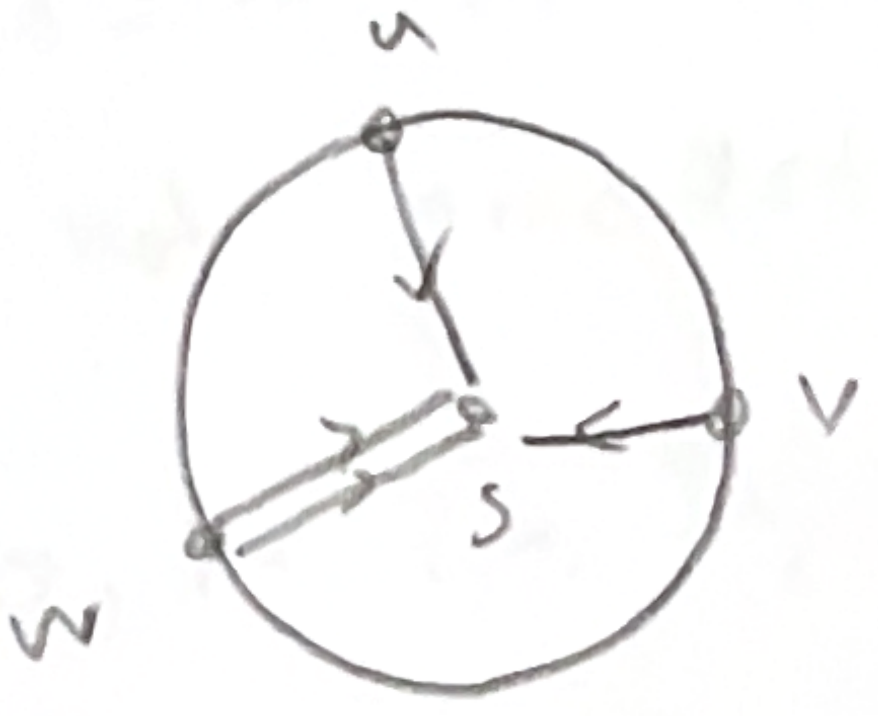
	0	1
0	0	1
1	1	1

$\{0, a\}$

$a + a = a$

Exercise for tutorial

① Describe the monoid of the wheel



Observe that $M \setminus \{0\}$ is a group.

② Describe the monoid of

on u 0, 1 chips

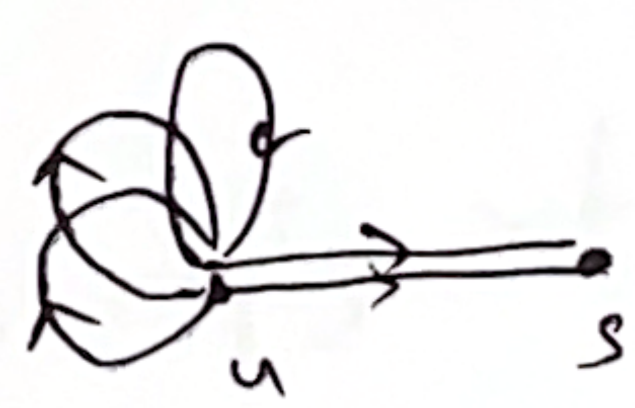
on v 0, 1, 2 chips

$ku + lv$ $\quad 0 \le k \le 1$

$0 \le l \le 2$

$|M| = 6$

Example for class



$0, 1u, 2u, 3u, 4u, \boxed{5u = 3u}$

	0	1u	2u	3u	4u
0	0	1u	2u	3u	4u
1u	1u	2u	3u	4u	3u
2u	2u	3u	4u	3u	4u
3u	3u	4u	3u	4u	3u
4u	4u	3u	4u	3u	4u

a couple of observations

$1u + 2u = 1u + 4u$ but $2u \neq 4u$

so not cancellative

$\{3u, 4u\} \subseteq \{0, 1u, 2u, 3u, 4u\}$ and

$\{3u, 4u\}$ is a group with $4u = 0$ of

the group $\cong \{0, 1\} \subseteq \mathbb{Z}_2$

the commutative monoid $M = \dfrac{\langle u \rangle}{\langle nu = mu \rangle}$

here $\boxed{3u = 5u}$.

Monoid — M a non-empty set

$+ : M \times M \longrightarrow M$ binary operation

which has a zero — $a + 0 = 0 + a = a$

commutative — $a + b = b + a$

associative — $(a+b)+c = a+(b+c)$

group ✓ — each element has an inverse

$\forall a, \exists b \quad a + b = b + a = 0.$

E sandpile

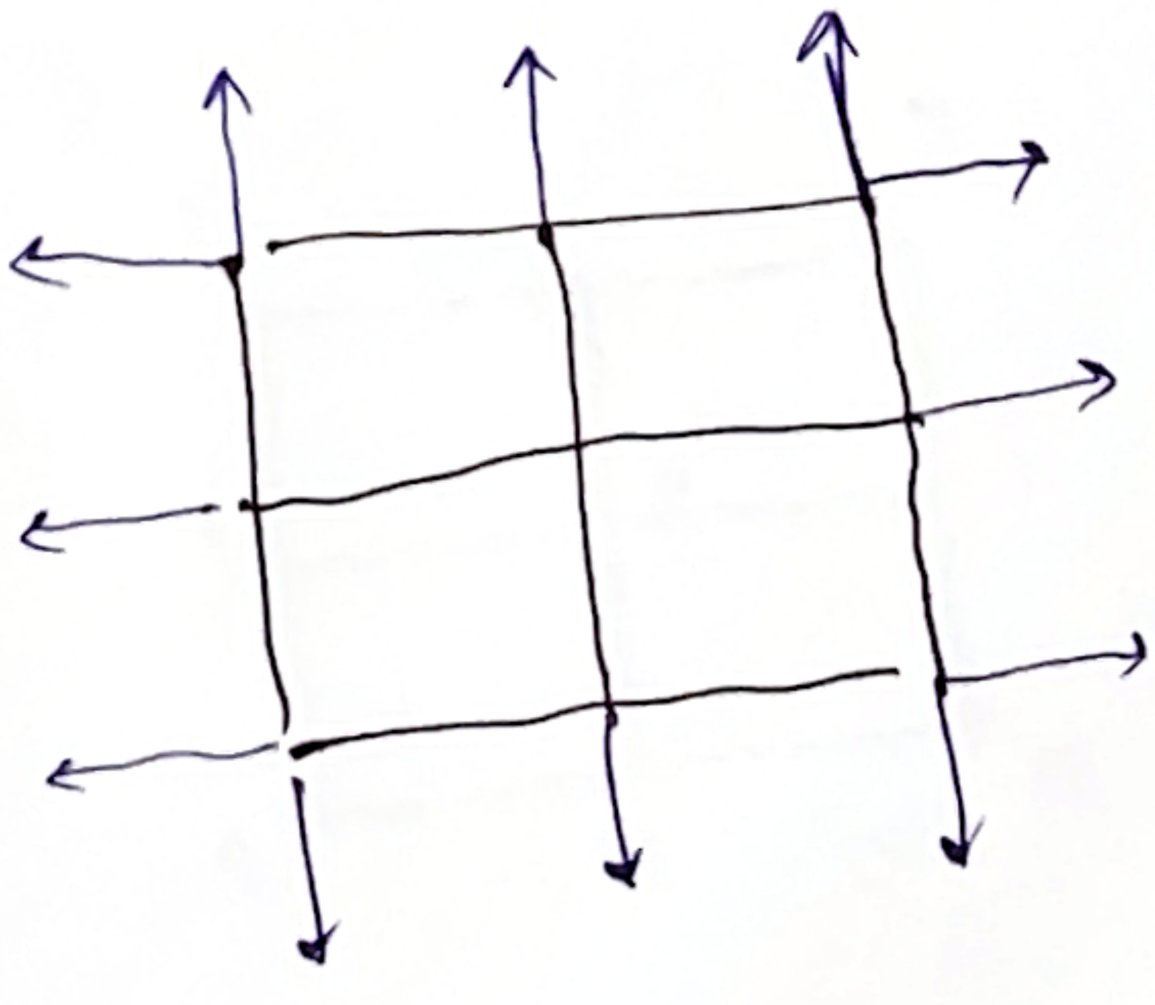
$SP(E)$ sandpile monoid

define $\left(\sum k_i u_i\right) + \left(\sum k_i' u_i\right) := \sum (k_i + k_i') u_i$

theme study $SP(E)$. does this reflect the behavior of sandpile model? (for example explain the fractal structure?)

Raising another (open) question

Consider



each node can have 4 configuration 0, 1, 2, 3.

So $4 \times 9 = 36$

$|SP(E)| = 36$

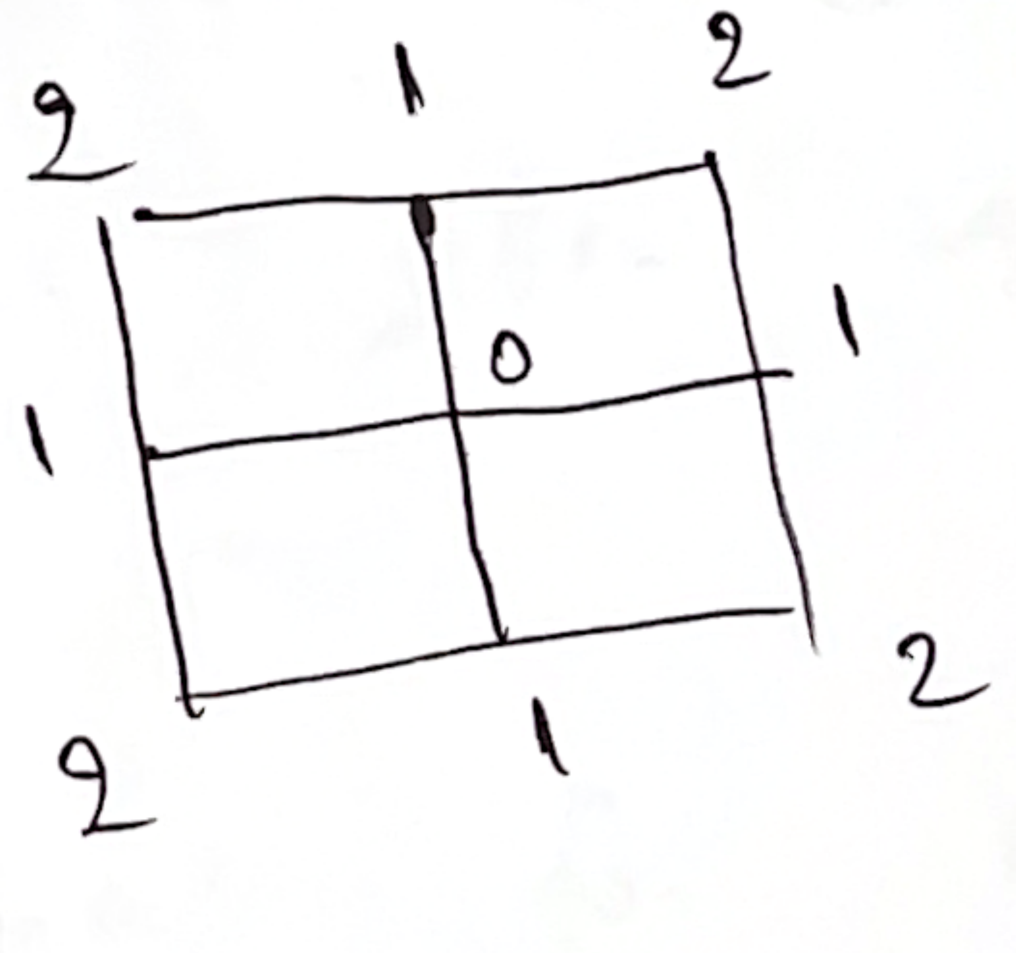
~~[illegible]~~

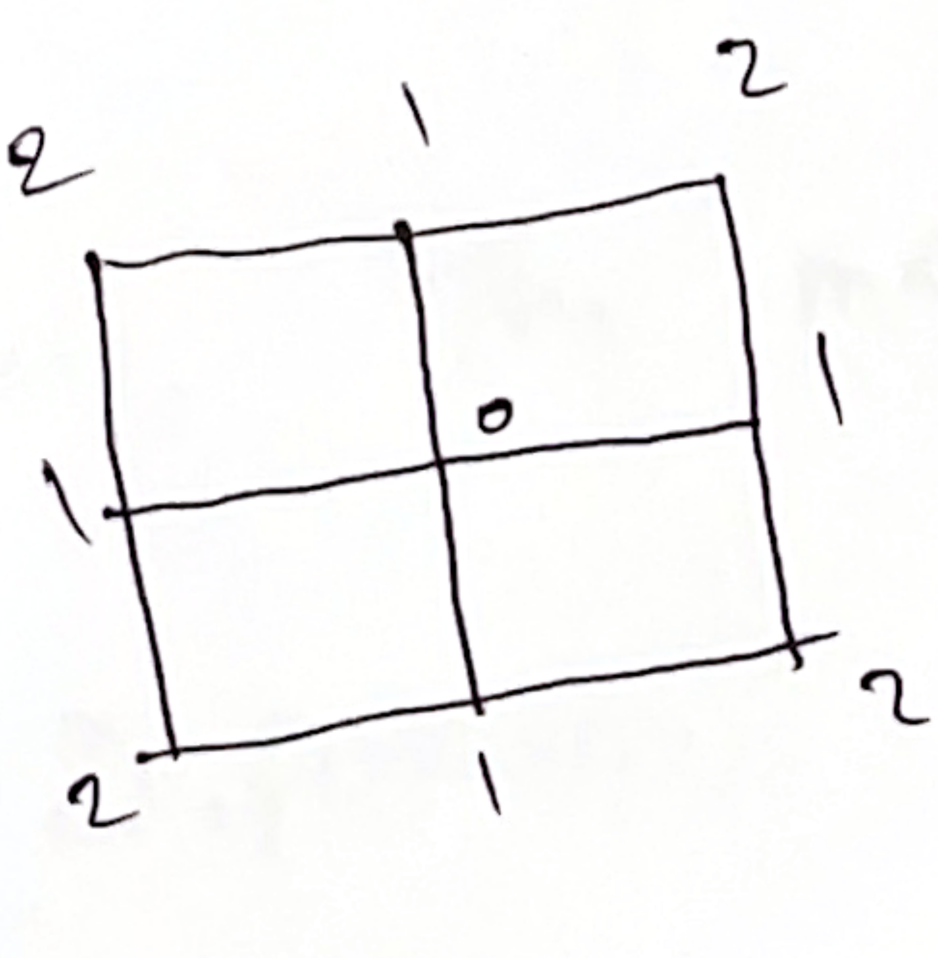
Similar to $M = \{0, 1_u, 2_u, 3_u, 4_u\}$ which contains $\{3_u, 4_u\}$ as a group with zero 4_u is it

this monoid also contains a "special" group.

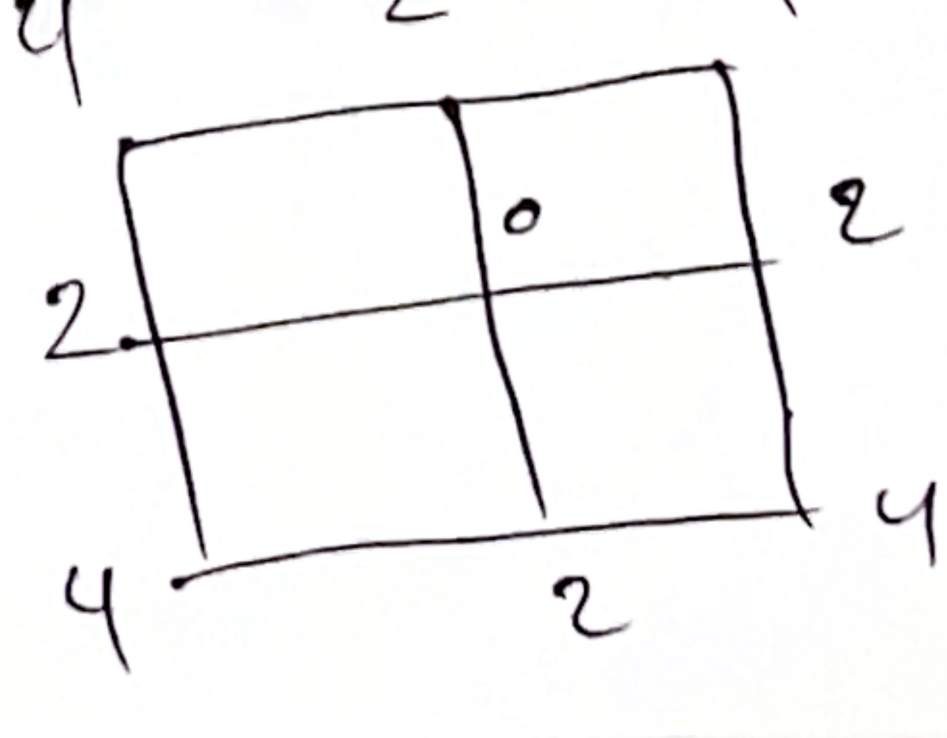
$$G(E) \subseteq SP(E)$$

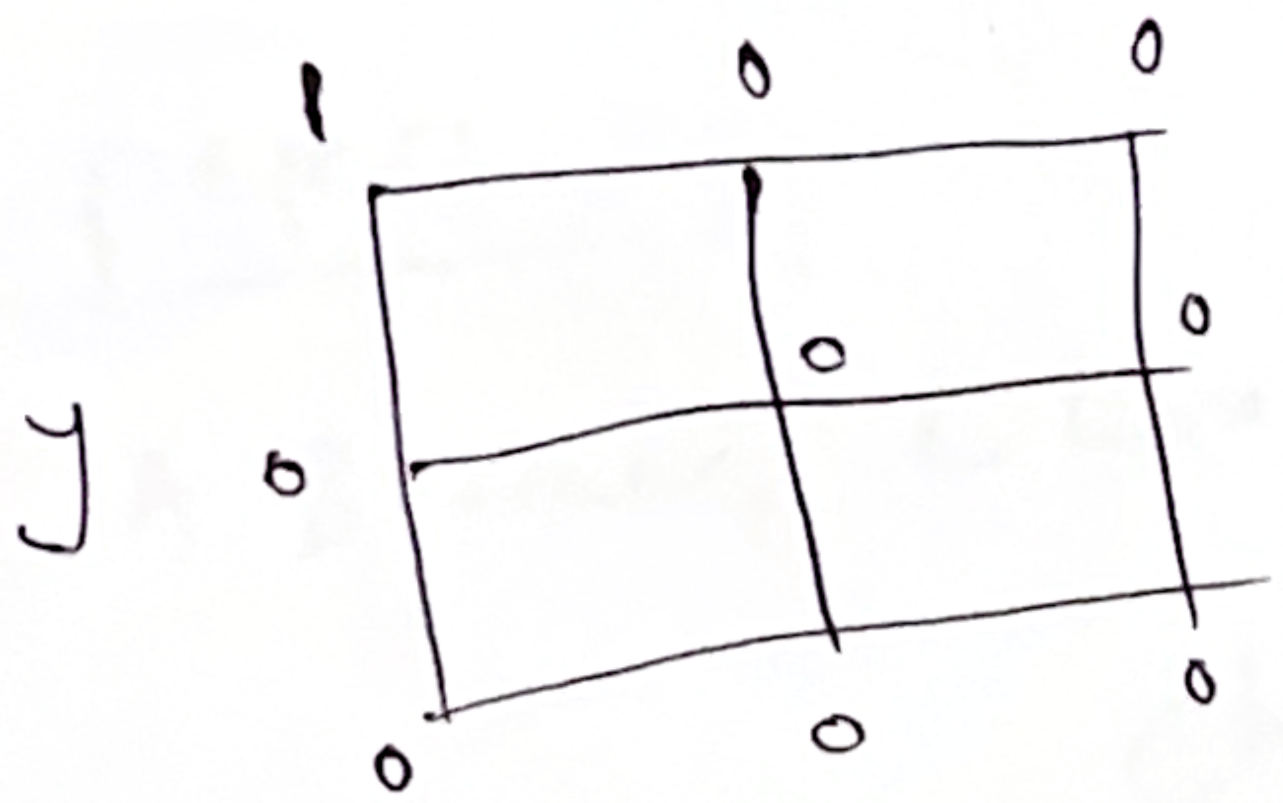
↑ sandpile group ↑ sandpile monoid, the zero of this group is

$x =$ 

Ex 

show $\boxed{x \oplus x = x}$

↪ $x \oplus x =$ 



$x + y \neq y$

we claimed $x = 0$ in the group. this shows $y \notin G(E)$.

29

Slide

explain why in Jew such a black of yellow appears?

References for chip firing

C. Klivans, The mathematics of chip-firing

S. Corry, D. Perkinson, Divisors and Sandpiles an Introduction to chip-firing